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On definable elements in models of PA

The aim of this paper is to describe the distribution of definable elements in models of PA. The analysis of definable elements proved to be fruitful in the context of investigating the relations between various types of induction, collection and minimalization. The classical results in this area were obtained by Kirby and Paris in (5); see also (4) for the discussion of parameter free induction. In addition, (2) and (3) can be recommended as a very useful, general reference. For more results in this direction, see (1).

We start with the following definition of structures consisting of elements of a model M definable by formulas belonging to Σ_n (Π_n).[1]

Definition 1.

$K^{\Sigma_n}(M) = \{a \in M : \exists\varphi(x) \in \Sigma_n\ \varphi(x) \text{ defines } a \text{ in } M\}$

$K^{\Pi_n}(M) = \{a \in M : \exists\varphi(x) \in \Pi_n\ \varphi(x) \text{ defines } a \text{ in } M\}$

The first useful fact concerns the relation between $Th_{\Pi_n}(N)$ (the set of all Π_n sentences true in the standard model) and $Th_{\Sigma_{n+1}}(N)$ (an analogous set of Σ_{n+1} sentences).

Fact 1. $\forall n \in N\ Th_{\Pi_n}(N) \vdash Th_{\Sigma_{n+1}}(N)$

Proof. Let $M \models Th_{\Pi_n}(N)$. Let φ be a Σ_{n+1} formula true in N of the form $\ulcorner \exists x\, \psi(x) \urcorner$. So for some n belonging to N, $N \models \psi(n)$. Since $\psi(n)$ belongs to Π_n, we have: $M \models \psi(n)$; therefore $M \models \varphi$.

□

The next two theorems state the sufficient and necessary conditions of the existence of Σ_n (or Π_n) definable, nonstandard elements in models of PA. In what follows, by „$K^{\Sigma_n}(M) > N$" we mean „there are nonstandard, Σ_n-definable elements of M"; analogously for $K^{\Pi_n}(M)$.[2]

Theorem 1. For every $n > 0$, for every $M \models$ PA, $K^{\Sigma_n}(M) > N$ iff $M \not\models Th_{\Pi_n}(N)$.

Proof. Assume at first, that $\varphi(x)$ is a Σ_n formula which defines in M a nonstandard element. Assume also that $M \models Th_{\Pi_n}(N)$. Then we have: $N \models \exists x\varphi(x)$, because otherwise $N \models \forall x \neg\varphi(x)$, so we would obtain: $M \models \forall x \neg\varphi(x)$, since this last formula belongs to Π_n (assuming that $n > 0$). So we obtain a certain $k \in$ N such that $N \models \varphi(k)$, but $M \not\models \varphi(k)$, since $\varphi(x)$ defines something nonstandard in M. However, $\varphi(k) \in \Sigma_n$, therefore $M \models \varphi(k)$ by our

[1] For the definition of the classes Σ_n and Π_n, see e.g. (2) or (3).

[2] Theorem 1 is folklore; we owe the idea of the proof of theorem 2 to H. Kotlarski and K. Zdanowski.

assumption that $M \models Th_{\Pi_n}(N)$ (cf. Fact 1). This is impossible, since $\varphi(x)$ was to define something nonstandard in M.

For the second implication, fix the smallest k such that $M \in Th_{\Pi_n}(N)$ (then $k \leq n$). Let $\psi = \ulcorner \forall x\varphi(x) \urcorner$ with $\varphi(x)$ belonging to Σ_{k-1}, such that $N \models \forall x\varphi(x)$ and $M \models \exists x\neg\varphi(x)$. Let $\tau(x)$ be a Σ_k formula which states that x is the smallest number satisfying $\neg\varphi(x)$. Then $\tau(x)$ defines something nonstandard in M, because if $M \models \tau(n)$ for some $n \in N$, then $M \models \neg\varphi(n)$, but $N \models \varphi(n)$, so $\varphi(n) \in Th_{\Sigma_{k-1}}(N)$. Therefore $M \not\models Th_{\Pi_{k-1}}(N)$, because otherwise $M \models \varphi(n)$. We obtain a contradiction, because k was to be the smaller number of this kind.

□

Theorem 2. For every $n \geq 0$, for every $M \models$ PA, $K^{\Pi_n}(M) > N$ iff M $\not\models Th_{\Pi_{n+1}}(N)$.

Proof.

(→) Assume that $\varphi(x)$ is a Π_n formula which defines in M a nonstandard element. Assume also that $M \models Th_{\Pi_{n+1}}(N)$. Then we have: $N \models \exists x\varphi(x)$, because otherwise $N \models \forall x\neg\varphi(x)$. Since this last formula is Π_{n+1}, we would have: $M \models \forall x\neg\varphi(x)$, which is impossible. So let $k \in N$ be such that $N \models \varphi(k)$. Therefore $M \models \varphi(k)$; but this is a contradiction, since $\varphi(x)$ was to define something nonstandard in M.

(←) Let n be the smallest natural number such that $M \not\models Th_{\Pi_{n+1}}(N)$. If $n = 0$, we have: $N \models \forall x\varphi(x)$ and M $\models \exists x\neg\varphi(x)$ for some formula $\varphi(x)$ belonging to Δ_0. Then the smallest x such that $\neg\varphi(x)$ is a nonstandard, Δ_0 definable element of M.

So now let's assume that $n > 0$. Let $N \models \forall x\exists y\neg\varphi(y, x)$ and $M \models \exists x\forall y\varphi(y, x)$ for φ belonging to Σ_{n-1}. In this case for every a, if $M \models \forall y\varphi(y, a)$, then $a > N$, because otherwise we would have for $k \in N$: $M \models \forall y\varphi(y, k)$. But $N \models \exists y\neg\varphi(y, k)$ and since this last formula is Σ_n it turns out that $M \not\models Th_{\Pi_n}(N)$, therefore $M \not\models Th_{\Pi_n}(N)$, which contradicts the choice of n.

Now take the following formula $\tau(z)$ belonging to Π_n:

$$\forall a, b[z = (a, b) \Rightarrow (\forall y\, \varphi(y, a) \wedge$$
$$\wedge\ \forall w < a\exists y < b\neg\varphi(y, w) \wedge$$
$$\wedge\ \forall c < b\exists w < a\forall y < c\, \varphi(y, w))]$$

So $\tau(z)$ states in effect, that z is a pair with the following properties:

(i) z is a pair whose first element a is a witness for $\ulcorner \exists x\, \forall y\varphi(y, x) \urcorner$

(ii) the second element b of z is an upper bound for counterexamples below a

(iii) b is the smallest such upper bound.

The first two conditions uniquely identify a, the last one uniquely identifies b, so in effect $\tau(z)$ is a Π_n formula defining a nonstandard element of M.

□

Our next result is a solution to the following problem: in which models of PA are there elements which are essentially Π_n-definable; we mean by that the elements which are Π_n, but not Σ_n definable.[3] In effect, our result is a partial
answer to the question if the Σ_n-Π_n hierarchy is strict.

Theorem 3. For every n, for every $M \models$ PA

If $M \not\models Th_{\Pi_{n+1}}(N)$, then for every $k \geq n+1$ $K^{\Sigma_k}(M) \subset K^{\Pi_k}(M)$.

Proof. Fix M, n and $k \geq n+1$. Let $\psi(x, y)$ be the following formula belonging to Π_k:

$$\forall\varphi \leq x[(\varphi \in \Pi_{k-1} \wedge \exists w Tr_{\Pi_{k-1}}(\varphi(w))) \Rightarrow \exists w < y Tr_{\Pi_{k-1}}(\varphi(w))] \wedge$$
$$\wedge \exists\varphi \leq x[\exists w \leq y Tr_{\Pi_{k-1}}(\varphi(w)) \wedge \forall w < y\text{-}1 \neg Tr_{\Pi_{k-1}}(\varphi(w))]$$

where $Tr_{\Pi_{k-1}}(x)$ is a universal formula for Π_{k-1} sentences. The formula $\psi(x, y)$ states in effect, that y is the least upper bound for witnesses for Π_{k-1} formulas below x. By Theorem 1, let $\gamma(x)$ be a Σ_k definition of some nonstandard element of M (here we use the assumption that $M \not\models Th_{\Pi_{n+1}}(N)$). Then the formula

$$\forall x[\gamma(x) \Rightarrow \psi(x, y)]$$

is a Π_k definition of some element a. Now assume that a is Σ_k definable by a formula $\ulcorner\exists z\chi(z, \cdot)\urcorner$. We take the following formula $\xi(w)$ belonging to Π_{k-1}:

$$\exists w_1,w_2 < w[w = (w_1,w_2) \wedge \chi(w_1,w_2)]$$

We have: $M \models \exists w\xi(w)$, so by our choice of a, $M \models \exists w < a\ \xi(w)$. But any such w would be a pair whose second element is a, since $\ulcorner\exists z\ \chi(z, \cdot)\urcorner$ defines a. This is impossible and we obtain a contradiction.

□

Note that the converse to Theorem 3 is false. As a counterexample, we may take $M \models Th_{\Pi_{n+1}}(N)$ such that $M \not\models Th_{\Pi_{n+2}}(N)$. Then by Theorem 2, $K^{\Pi_{n+1}}(M) > N$, but by Theorem 1, $K^{\Sigma_{n+1}}(M) = N$, so $K^{\Sigma_{n+1}}(M) \subset K^{\Pi_{n+1}}(M)$. By Theorem 3, we have also $K^{\Sigma_k}(M) \subset K^{\Pi_k}(M)$ for all $k \geq n+2$. So the subsequent of Theorem 3 holds, but the antecedent is obviously false.

In a similar vein, we consider now the question: are there Σ_{n+1} definable elements which are not Π_n definable. Together with the previous result our next theorem gives us in effect the strict Σ_n-Π_n hierarchy. However, Theorem 4 contains also some extra information: there is always an essentially Σ_{n+1} definable element below an arbitrarily chosen Σ_{n+1} definable one.

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[3] Note that a Σ_n-definable element is always Π_n-definable, because if $\varphi(x)$ is Σ_n and defines a, then $\ulcorner\forall x[\varphi(x) \Rightarrow y = x]\urcorner$ is Π_n and defines a.

Theorem 4. Let $M \models$ PA, let $n \in N$ and let $a \in M$ such that $a > N$ and $a \in K^{\Sigma_{n+1}}(M)$. Then for some $b < a$, $b \in K^{\Sigma_{n+1}}(M)$ and $b \notin K^{\Pi_n}(M)$.

Proof. Fix an appropriate *M*, *n* and *a*. Let $\psi(z) \in \Sigma_{n+1}$ and define *a*. Consider the following formula $\chi(s)$ belonging to Σ_{n+1}:

$$\exists z\, [\psi(z) \wedge \forall \varphi < z(\varphi \in \Pi_n \Rightarrow (\neg Tr_{\Pi_n}(\varphi\,(s)) \vee \exists w < s\ Tr_{\Pi_n}(\varphi\,(w)))) \wedge$$
$$\wedge\ \forall w < s \exists \varphi < z[\varphi \in \Pi_n \wedge Tr_{\Pi_n}(\varphi\,(w)) \wedge \forall x < w\, \neg Tr_{\Pi_n}(\varphi\,(x))]$$

Intuitively, the formula $\chi(s)$ states that:

1. *s* is not the smallest object satisfying some Π_n formula below *a*
2. *s* is the smallest object with the above property.

Now we are going to show that $\chi(s)$ defines something below *a*. Let $\alpha(\varphi, s)$ be the formula:

$$\neg Tr_{\Pi_n}(\varphi\,(s)) \vee \exists w < s\ Tr_{\Pi_n}(\varphi\,(w))$$

So the formula $\alpha(\varphi, s)$ states that φ doesn't identify *s*.
In order to see that $\chi(s)$ defines something below *a*, it is clearly enough to observe that:

$$\text{PA} \vdash \forall z \exists s \le z \forall \varphi < z[\varphi \in \Pi_n \Rightarrow \alpha(\varphi, s)]$$

For an indirect proof, fix *z* in a model *M* of PA such that in *M*:

$$\forall s \le z \exists \varphi < z[\varphi \in \Pi_n \wedge \neg\alpha(\varphi, s)]$$

From pigeonhole principle we obtain:

$$\exists s_1, s_2 \le z \exists \varphi < z[s_1 \ne s_2 \wedge \varphi \in \Pi_n \wedge \neg\alpha(\varphi, s_1) \wedge \neg\alpha(\varphi, s_2)]$$

But this means that both s_1 and s_2 are the smallest numbers satisfying φ, but nonetheless they are different. Since this is impossible, it ends the proof.

□

Now we are ready to consider in general terms the question concerning the distribution of definable elements in models of PA. In accordance with the usual convention, we write "$\subseteq_e$" ("$\subseteq_{cf}$") to denote an end (cofinal) extension of a given structure.

Theorem 5. For every *n*, for every $M \models$ PA:

(a) $K^{\Sigma_n}(M) \subseteq_e K^{\Pi_n}(M)$

(b) $K^{\Pi_n}(M) \subseteq_{cf} K^{\Sigma_{n+1}}(M)$

Proof.

(a) Let $\varphi(x) \in \Sigma_n$ and define a. Then the formula: $\ulcorner \forall x[\varphi(x) \Rightarrow x = y] \urcorner$ is Π_n and defines a. This shows that $K^{\Sigma_n}(M) \subseteq K^{\Pi_n}(M)$. As for being an initial segment, let $\varphi(x) \in \Pi_n$ and define a; let $\psi(x) \in \Sigma_n$ and define b. Assume also that $a < b$. Then a is Σ_n definable by the formula:

$$\exists x\, [\psi(x) \wedge \forall w < x(\varphi\, (w) \Rightarrow w = y)]$$

(b) The inclusion itself is obvious. So we show only that there is a Π_n definable element above each Σ_{n+1} definable one.

Let $a \in K^{\Sigma_{n+1}}(M)$. Let $\psi(x)$ be a Σ_{n+1} definition of a. If $n = 0$, then $\psi(x)$ has a form $\ulcorner \exists y \varphi\, (y, x) \urcorner$ for φ belonging to Δ_0. In this case the formula "w is the smallest pair such that $\varphi(w_0, w_1)$" is Δ_0 and defines a number greater than a.

So now we may assume that $n > 0$ and our formula $\psi(x)$ has the form: $\ulcorner \exists s \forall y \varphi(y, s, x) \urcorner$. Then $M \models \exists x[x = (x_0, x_1) \wedge \forall y \varphi(y, x_0, x_1)]$. Now let $\gamma(z)$ be the following formula:

$$\exists c, d < z[z = (c, d) \wedge \forall y \varphi\, (y,\ c_0, c_1) \wedge$$
$$\forall w < c \exists y < d\, \neg\varphi\, (y, w_0, w_1) \wedge$$
$$\forall e < d \exists w < c \forall y < e \varphi\, (y, w_0, w_1)]$$

(In our notation e.g. "w_0" denotes the left element of the pair w).

This formula defines a number z, which is a pair (c, d), where $c = (\cdot, a)$, therefore $z > a$. In addition, $z \in K^{\Pi_n}(M)$, because $\gamma(z)$ is a Π_n formula.

□

The following corollary states the sufficient condition under which the above inclusions are proper.

Corollary 1. Let $M \not\models Th_{\Pi_{n+1}}(N)$. Then:

(a) $K^{\Sigma_{n+1}}(M) \subset_e K^{\Pi_{n+1}}(M)$

(b) $K^{\Pi_n}(M) \subset_{cf} K^{\Sigma_{n+1}}(M)$

Proof.

(a) By Theorem 5a, $K^{\Sigma_{n+1}}(M) \subseteq_e K^{\Pi_{n+1}}(M)$. Since $M \not\models Th_{\Pi_{n+1}}(N)$, by Theorem 3 we obtain: $K^{\Sigma_{n+1}}(M) \subset_e K^{\Pi_{n+1}}(M)$, which is our desired result.

(b) We have: $K^{\Pi_n}(M) \subseteq_{cf} K^{\Sigma_{n+1}}(M)$ by Theorem 5b. Since $M \not\models Th_{\Pi_{n+1}}(N)$, we obtain $K^{\Sigma_{n+1}}(M) > N$ by Theorem 1. So by Theorem 4, $K^{\Pi_n}(M) \subset_{cf} K^{\Sigma_{n+1}}(M)$. And that ends the proof.

□

Corollary 2 formulated below is a negative result. It states that no interesting relations (no relations of being an initial segment or having a cofinal extension) hold between Π_n-type structures, unless in the standard case.

Corollary 2.

(a) $K^{\Pi_n}(M) \subseteq_e K^{\Pi_{n+1}}(M)$ iff $K^{\Pi_n}(M) = N$

(b) $K^{\Pi_n}(M) \subseteq_{cf} K^{\Pi_{n+1}}(M)$ iff $K^{\Pi_{n+1}}(M) = N$

Proof.

(a) The implication from right to left is obvious. So let's assume that $K^{\Pi_n}(M) \neq N$. In this case by Theorem 2, $M \nvDash Th_{\Pi_{n+1}}(N)$. So by Corollary 1b, there is an $a \in K^{\Sigma_{n+1}}(M)$ such that $a \notin K^{\Pi_n}(M)$. Therefore $a \in K^{\Pi_{n+1}}(M)$. But $K^{\Pi_n}(M) \subseteq_{cf} K^{\Sigma_{n+1}}(M)$ (Theorem 5b), so there is a b belonging to $K^{\Pi_n}(M)$ such that $b > a$. Therefore it is not the case that $K^{\Pi_n}(M) \subseteq_e K^{\Pi_{n+1}}(M)$.

(b) The implication from right to left is obvious. So let's assume that $K^{\Pi_n}(M) \subseteq_{cf} K^{\Pi_{n+1}}(M)$ and that $K^{\Pi_{n+1}}(M) \neq N$. So $K^{\Pi_n}(M) \neq N$. By Theorem 2, $M \nvDash Th_{\Pi_{n+1}}(N)$. Therefore by Corollary 1a, $K^{\Sigma_{n+1}}(M) \subset_e K^{\Pi_{n+1}}(M)$, so we obtain: not $K^{\Pi_n}(M) \subseteq_{cf} K^{\Pi_{n+1}}(M)$, which ends the proof, producing the desired contradiction.

□

As we already remarked, structures on Σ_n-definable elements proved to be useful in analysing the relations between various types of induction. A typical question worth considering in this context was: how much induction satisfies a given Σ_n-structure. Obviously one can ask the same question about Π_n-structures. However, the next lemma leads to a negative conclusion: such structures don't even satisfy PA^- (arithmetic without induction scheme). On a more positive vein, the lemma will permit us to characterize $K^{\Sigma_{n+1}}(M)$ as the closure of $K^{\Pi_n}(M)$ under subtraction (which is Corollary 3).

Lemma 1. For every $n \geq 0$, for every $w \in K^{\Sigma_{n+1}}(M)$, there is an $a \in K^{\Pi_n}(M)$ such that $w+a \in K^{\Pi_n}(M)$.

Proof. Fix $w \in K^{\Sigma_{n+1}}(M)$. Let $\ulcorner \exists s \forall y \varphi(y, s, x) \urcorner$ be a Σ_{n+1} definition of w. Pick an object a as in the proof of Theorem 5b - then a is a pair (c, d) such that (c_0, c_1) satisfy $\ulcorner \forall y \varphi(y, s, x) \urcorner$ and d is the smallest upper bound for counterexamples below c. Such an a belongs to $K^{\Pi_n}(M)$. Then $w+a$ has the following Π_n definition:

$$\exists xy < s[x = a \wedge \exists z < x(z = a_0 \wedge y = z_1 \wedge s = x + y)]$$

Remember that $\ulcorner x = a \urcorner$ can be written down as a Π_n formula without parameters. As usual, we write a_0 and a_1 to denote the left and the right member of the pair a.

□

Corollary 3. $K^{\Sigma_{n+1}}(M)$ is exactly the closure of $K^{\Pi_n}(M)$ under subtraction.

Proof. Obvious, from Lemma 1.

□

It follows in particular that for every $n \geq 0$, $K^{\Pi_n}(M) \not\models$ PA$^-$. Take simply an object w belonging to $K^{\Sigma_{n+1}}(M)$ such that $w \notin K^{\Pi_n}(M)$. By Lemma 1, fix an a belonging to $K^{\Pi_n}(M)$ such that $w+a \in K^{\Pi_n}(M)$. Therefore in $K^{\Pi_n}(M)$ $w+a$ is not a sum of a with any other number.

Our next result is an addition to Theorem 4. We showed before that there is an essentially Σ_{n+1} definable element below each Σ_n definable one. Now we consider a special case of a Π_n definable element a greater that $K^{\Sigma_n}(M)$. We show that such an a determines an essentially Σ_{n+1}-definable b which similarily will be greater than $K^{\Sigma_n}(M)$.

In te proof we will use the following fact (we write "$I^n(M)$" to denote an initial segment of M cofinal with $K^{\Sigma_n}(M)$).[4]

Fact 2. For every $n \geq 1$, $I^n(M) \models Th_{\Pi_{n+1}}(M)$
For the proof, see (3).[5]

Theorem 6. Let $n \geq 1$. For every $a \in K^{\Pi_n}(M)$, if $a > K^{\Sigma_n}(M)$, then there is a number $b < a$ such that $b \in K^{\Sigma_{n+1}}(M)$ and $b > K^{\Sigma_n}(M)$ and $b \notin K^{\Pi_n}(M)$.

Proof. Fix $a \in K^{\Pi_n}(M)$ such that $a > K^{\Sigma_n}(M)$. Let s be the smallest number such that no Π_n formula below a identifies s (in other words, let s satisfy the formula $\chi(s)$ used in the proof of Theorem 4). So $s < a$, $s \in K^{\Sigma_{n+1}}(M)$ and $s \notin K^{\Sigma_n}(M)$. We now claim that $s > K^{\Sigma_n}(M)$. By Fact 2, $I^n(M) \models Th_{\Pi_{n+1}}(M)$. Therefore for $n \geq 1$:

$$I^n(M) \models \forall x \exists \varphi\, [\varphi \in \Delta_0 \wedge Tr_0(\ulcorner \varphi(x) \urcorner) \wedge \forall w < x \neg Tr_0(\ulcorner \varphi(w) \urcorner)]$$

It follows that $s > I^n(M)$, therefore $s > K^{\Sigma_n}(M)$. Otherwise we would have:

$$M \models \exists \varphi < a[\varphi \in \Delta_0 \wedge s \text{ is the smallest object satisfying } \varphi]$$

But this is impossible by the choice of s.

□

References

[4] Formally: $I^n(M) = \{a \in M: \exists b \in K^{\Sigma_n}(M)\ M \models a \leq b\}$.
[5] This is Theorem 10.8 in Kaye's book.

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