Introduction to formal theories of truth, homework

1 Notation

- PA is Peano arithmetic; L_{PA} is the language of Peano arithmetic, $Sent_{L_{PA}}$ is the set of sentences of L_{PA}.
- The language L_T is L_{PA} extended with the predicate '$T(x)$'; $Sent_{L_T}$ is the set of sentences of L_T.
- Tm^c is the set of constant terms of L_{PA} (terms which do not contain any variables).
- '$val(t) = x$' is an arithmetical formula with the intuitive reading 'the value of the term t is x'.
- $\models_{sk}$ is the strong Kleene satisfaction relation (defined in the lecture).

2 Definitions

Definition 1 (Theory CT^-) *Apart from all the axioms of PA, CT^- has the following truth axioms:*

(1) $\forall s, t \in Tm^c\Big(T(s = t) \equiv val(s) = val(t)\Big)$

(2) $\forall x\Big(Sent_{L_{PA}}(x) \rightarrow (T\neg x \equiv \neg Tx)\Big)$

(3) $\forall x \forall y\Big(Sent_{L_{PA}}(x \wedge y) \rightarrow (T(x \wedge y) \equiv (Tx \wedge Ty))\Big)$

(4) $\forall v \forall x\Big(Sent_{L_{PA}}(\forall v \psi) \rightarrow (T(\forall v \psi) \equiv \forall x T(\psi(x/v)))\Big)$

Definition 2 (Kripke's construction)

- $T_0^+ = T_0^- = \emptyset$,
  $N_0 = (N, T_0^+, T_0^-)$;
- $T_{\alpha+1}^+ = \{\psi : N_\alpha \models_{sk} \psi\}$,
  $T_{\alpha+1}^- = \{\psi : N_\alpha \models_{sk} \neg\psi\}$,
  $N_{\alpha+1} = (N, T_{\alpha+1}^+, T_{\alpha+1}^-)$.

For λ being a limit ordinal, we define:

- $T^+_\lambda = \bigcup_{\alpha<\lambda} T^+_\alpha$,
  $T^-_\lambda = \bigcup_{\alpha<\lambda} T^-_\alpha$,
  $N_\lambda = (N, T^+_\lambda, T^-_\lambda)$.

Definition 3 (Theory KF) *KF is a theory obtained from PAT by adding the following truth axioms:*

(1) $\forall s \forall t \Big(T(s = t) \equiv val(s) = val(t)\Big)$

(2) $\forall s \forall t \Big(T(\neg s = t) \equiv val(s) \neq val(t)\Big)$

(3) $\forall x \Big(Sent_{L_T}(x) \to (T(\neg\neg x) \equiv Tx)\Big)$

(4) $\forall x \forall y \Big(Sent_{L_T}(x \wedge y) \to (T(x \wedge y) \equiv Tx \wedge Ty)\Big)$

(5) $\forall x \forall y \Big(Sent_{L_T}(x \wedge y) \to (T\neg(x \wedge y) \equiv T\neg x \vee T\neg y)\Big)$

(6) $\forall x \forall y \Big(Sent_{L_T}(x \vee y) \to (T(x \vee y) \equiv Tx \vee Ty)\Big)$

(7) $\forall x \forall y \Big(Sent_{L_T}(x \vee y) \to (T\neg(x \vee y) \equiv T\neg x \wedge T\neg y)\Big)$

(8) $\forall v \forall x \Big(Sent_{L_T}(\forall vx) \to (T(\forall vx) \equiv \forall t T(x(t/v)))\Big)$

(9) $\forall v \forall x \Big(Sent_{L_T}(\forall vx) \to (T(\neg\forall vx) \equiv \exists t T(\neg x(t/v)))\Big)$

(10) $\forall v \forall x \Big(Sent_{L_T}(\exists vx) \to (T(\exists vx) \equiv \exists t T(x(t/v)))\Big)$

11) $\forall v \forall x \Big(Sent_{L_T}(\exists vx) \to (T(\neg\exists vx) \equiv \forall t T(\neg x(t/v)))\Big)$

(12) $\forall t (T(Tt) \equiv T(val(t))$

(13) $\forall t \Big(T\neg Tt \equiv (T(\neg val(t)) \vee \neg Sent_{L_T}(val(t)))\Big)$

Definition 4 (The axioms of consistency and completeness)

CONS $\quad \forall x \Big(Sent_{L_T}(x) \to \neg(Tx \wedge T\neg x)\Big)$

COMPL $\quad \forall x \Big(Sent_{L_T}(x) \to (Tx \vee T\neg x)\Big)$

3 Tasks

Choose one of the following tasks.

1. Prove that for every $\varphi(x_0 \dots x_n) \in L_{PA}$:

$$CT^- \vdash \forall x_0 \dots x_n \Big[T(\varphi(x_0 \dots x_n)) \equiv \varphi(x_0 \dots x_n)\Big].$$

2. Complete the inductive part of the proof of the following lemma (N_α is as in Definition 2):

$$\forall\alpha\forall\varphi \in Sent_{L_T}\Big[\text{if } N_\alpha \models_{sk} T(\varphi),\ \text{then } N_\alpha \models_{sk} \varphi\Big].$$

3. Let κ be a fixed point of Kripke's construction (see Definition 2) and let N^{Cl} be defined as (N, T^+_κ). Choose four axioms of KF + CONS and demonstrate that they are true in N^{Cl}.

4. Let κ be a fixed point of Kripke's construction and let N^d be defined as (N, T^d), where $T^d = \{\varphi \in Sent_{L_T} : \varphi \notin T^-_\kappa\}$. Choose four axioms of KF + COMPL and demonstrate, that they are true in N^d.