Yablo's paradox in axiomatic theories of truth

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Yablo's paradox

Consider an infinite sequence of sentences $Y_0, Y_1, Y_2, \ldots$ such that:

$$\begin{aligned} Y_0 &= \forall z > 0\, \neg T(Y_z), \\ Y_1 &= \forall z > 1\, \neg T(Y_z), \\ Y_2 &= \forall z > 2\, \neg T(Y_z), \\ &\vdots \end{aligned}$$

Assume Y_k is true. Then for any $i > k$, Y_i is not true, and in particular Y_{k+1} is not true. But also $\forall z > k+1\, Y_z$ is not true, so Y_{k+1}, therefore Y_{k+1} is true after all - a contradiction. Since the reasoning goes for an arbitrary k, we obtain: all Y_k-s are not true. But then Y_0 is true - a contradiction.

Yablo formulas and sentences

Definition

- $Y(x)$ is a Yablo formula in a theory S with respect to the predicate $T(x)$ of the language of S iff it satisfies (provably in S) the *Yablo condition*, i.e. iff

$$S \vdash \forall x[Y(x) \equiv \forall z > x \neg T(Y(z))]$$

- φ is a Yablo sentence in S iff φ is obtained by substituting a numeral for x in $Y(x)$.

Existence of Yablo formulas

Theorem

For every theory S in L_T extending Robinson's arithmetic, there is a Yablo formula in S.

The theorem follows from the diagonal lemma in the following form:

Lemma

Let S be a theory in L_T extending Robinson's arithmetic. Then for every $\varphi(x, y) \in L_T$ there is a formula $\psi(x)$ such that:

$$S \vdash \psi(x) \equiv \varphi(x, \ulcorner \psi(x) \urcorner)$$

Proof of the existence theorem

Proof of the theorem.

Fix:

$$\varphi(x, y) := \forall z > x\ \neg T(sub(y, name(z))).$$

By the diagonal lemma, take $Y(x)$ such that:

$$S \vdash Y(x) \equiv \forall z > x\ \neg T(sub(\ulcorner Y(x) \urcorner, name(z))).$$

$\square$

Questions

Question 1 Which Yablo sentences are provable/disprovable in a given truth theory?

Question 2 Are all Yablo sentences provably equivalent in a given theory?

Question 3 Are Yablo sentences equivalent (provably in a given theory) to statements of their own untruth?

Question 4 To what extent does the answer to Questions 1-3 depend on our choice of a Yablo formula $Y(x)$?

The simplest case: *PAT*

Fact

Let $Y(x)$ be a Yablo formula in PAT. Then:

- **(a)** $PAT \nvdash \exists x Y(x)$
- **(b)** $PAT \nvdash \exists x \neg Y(x)$
- **(c)** *If $Y(x)$ contains a free variable x, then for all natural numbers n and k, if $n > k$, then* $PAT \nvdash Y(n) \rightarrow Y(k)$

Proof.

Since T in PAT functions just as a new predicate, $PAT \nvdash \exists x T(x)$ and also $PAT \nvdash \exists x \neg T(x)$, therefore both (a) and (b) follow trivially. For (c), assume that $Y(x)$ contains a free variable x. Then for every n and k, if $n \neq k$, then $\ulcorner Y(k) \urcorner \neq \ulcorner Y(n) \urcorner$. Consider a model (N, T) with $T = \{Y(n)\}$. Then we have: $(N, T) \models Y(n)$; $(N, T) \nvDash Y(k)$. □

Friedman-Sheard system

FS is defined as the system with the following axioms and rules:

1. $\forall s, t \in Tm^c \big(T(s{=}t) \equiv val(s){=}val(t) \big)$
2. $\forall x \, \big(\mathrm{Sent}_T(x) \rightarrow (T\neg x \equiv \neg Tx) \big)$
3. $\forall x \, \forall y \, \big(\mathrm{Sent}_T(x \wedge y) \rightarrow (T(x \wedge y) \equiv (Tx \wedge Ty)) \big)$
4. $\forall x \, \forall y \, \big(\mathrm{Sent}_T(x \vee y) \rightarrow (T(x \vee y) \equiv (Tx \vee Ty)) \big)$
5. $\forall v \, \forall x \, \big(\mathrm{Sent}_T(\forall vx) \rightarrow (T(\forall vx) \equiv \forall t \, T(x(t/v))) \big)$
6. $\forall v \, \forall x \, \big(\mathrm{Sent}_T(\exists vx) \rightarrow (T(\exists vx) \equiv \exists t \, T(x(t/v))) \big)$

$$NEC \quad \frac{\phi}{T\phi} \qquad\qquad \frac{T\phi}{\phi} \quad CONEC$$

Yablo formulas in *FS*

Fact

$FS^{-} \vdash \forall xz[x < z \rightarrow (Y(x) \rightarrow Y(z))]$

Corollary

$\forall x, z[x < z \rightarrow (T(Y(x)) \rightarrow T(Y(z))]$

Yablo formulas in *FS*

Theorem

$FS^{-} \vdash \forall xz[Y(x) \equiv Y(z)]$.

Proof.

Assume $z > x$. Then $Y(x) \rightarrow Y(z)$. In the argument for the opposite implication:

1. Assume $Y(z)$ and $\neg Y(x)$
2. Then $\forall s > z \neg T(Y(s))$ and $\exists s > x T(Y(s))$. In particular, $\neg T(Y(z+1))$
3. Fix s such that $s \leq z \wedge T(Y(s))$
4. By Corollary, $T(Y(z+1))$ - a contradiction.

□

Yablo formulas in *FS*

Corollary

$FS^{-} \vdash \forall xz[T(Y(x)) \equiv T(Y(z))].$

The proof is immediate, by applying NEC and compositional axioms to the previous Theorem. We have also:

Corollary

$FS^{-} \vdash \forall x[Y(x) \equiv \neg T(Y(x))].$

In effect each Yablo sentence is a liar. Finally we obtain:

Fact

If FS is consistent, then:

(a) $FS \nvdash \exists x Y(x)$

(b) $FS \nvdash \exists x \neg Y(x)$

The theory *KF*

1. $\forall s \forall t \, (T(s = t) \equiv val(s) = val(t))$
2. $\forall s \forall t \, (T(\neg s = t) \equiv val(s) \neq val(t))$
3. $\forall x \, (\mathrm{Sent}_T(x) \rightarrow (T(\neg\neg x) \equiv Tx))$
4. $\forall x \forall y \, (\mathrm{Sent}_T(x \wedge y) \rightarrow (T(x \wedge y) \equiv Tx \wedge Ty))$
5. $\forall x \forall y \, (\mathrm{Sent}_T(x \wedge y) \rightarrow (T\neg(x \wedge y) \equiv T\neg x \vee T\neg y))$

6-7 Similarly for disjunction

8. $\forall v \forall x \, (\mathrm{Sent}_T(\forall vx) \rightarrow (T(\forall vx) \equiv \forall t \, T(x(t/v))))$
9. $\forall v \forall x \, (\mathrm{Sent}_T(\forall vx) \rightarrow (T(\neg\forall vx) \equiv \exists t \, T(\neg x(t/v))))$

10-11 Similarly for the existential quantifier

12. $\forall t \, (T(Tt) \equiv T(val(t)))$
13. $\forall t \, (T\neg Tt \equiv (T\neg val(t) \vee \neg \mathrm{Sent}_T(val(t))))$

Additional axioms

Consistency

$$\textbf{(Cons)} \qquad \forall x \left(\mathrm{Sent}_T(x) \rightarrow \neg(Tx \wedge T\neg x) \right)$$

Completeness

$$\textbf{(Compl)} \qquad \forall x \left(\mathrm{Sent}_T(x) \rightarrow (Tx \vee T\neg x) \right)$$

Truth introduction and elimination

Fact

For every $\varphi(x_1 ... x_n)$:

(T-out) $KF + Cons \vdash \forall x_1 ... x_n [T(\varphi(x_1 ... x_n)) \rightarrow \varphi(x_1 ... x_n)]$

(T-in) $KF + Compl \vdash \forall x_1 ... x_n [\varphi(x_1 ... x_n)) \rightarrow T(\varphi(x_1 ... x_n))]$

The fact is proved by induction on complexity of φ. The following corollary can be obtained:

Corollary

Let L be such that $KF \vdash L \equiv \neg T(L)$. Then:

1. $KF + Cons \vdash L$
2. $KF + Compl \vdash \neg L$

KF - definitions and facts

Definition

For $(M, T) \models KF$, we denote:

- $T^+ = T$
- $T^- = \{z : \neg z \in T^+\}$
- $M^* = (M, T^+, T^-)$

Definition

- $M^* \models_{sk} s = t$ iff $val(s) = val(t)$; similarly for negation.
- $M^* \models_{sk} Tt$ iff $val(t) \in T^+$.
- $M^* \models_{sk} \neg Tt$ iff $(Sent(val(t))$ and $val(t) \in T^-)$ or $\neg Sent(val(t))$.
- $M^* \models_{sk} \neg\neg\varphi$ iff $M^* \models_{sk} \varphi$.
- $M^* \models_{sk} \varphi \wedge \psi$ iff $M^* \models_{sk} \varphi$ and $M^* \models_{sk} \psi$.
- $M^* \models_{sk} \neg(\varphi \wedge \psi)$ iff $M^* \models_{sk} \neg\varphi$ or $M^* \models_{sk} \neg\psi$.
- Similarly for disjunction and its negation.
- $M^* \models_{sk} \forall x \varphi(x)$ iff for all $a \in M$ $M^* \models_{sk} \varphi(a)$.
- $M^* \models_{sk} \neg\forall x \varphi(x)$ iff for some $a \in M$ $M^* \models_{sk} \neg\varphi(a)$.

KF-truth is well behaved

Theorem

If $(M, T) \models KF$*, then* $\forall \varphi \in L_T$ $[M^* \models_{sk} \varphi$ *iff* $M^* \models T(\varphi)]$.

Proof.

E.g. for $\varphi = \neg T(t)$ we have: $M^* \models_{sk} \neg T(t)$ iff $t \in T^- \vee \neg Sent(t)$ iff $\neg t \in T^+ \vee \neg Sent(t)$ iff $(M, T) \models T(\neg t) \vee \neg Sent(t)$ iff $(M, T) \models T(\neg T(t))$ iff $\neg T(t) \in T^+$ iff $M^* \models_{sk} T(\neg T(t))$. In the inductive part, we reason by induction on positive complexity of φ.

$\square$

Dual models

Definition

For $(M, T) \models KF$, we define:

- $T^d = Sent - T^-$
- $M^d = (M, T^d)$

Theorem

(a) *If* $(M, T) \models KF$*, then* $(M, T^d) \models KF1 - KF12$

(b) *If* $(M, T) \models KF + Cons$*, then* $(M, T^d) \models KF + Compl$

Yablo formulas in $KF + Cons$

Theorem

For every natural number n, there are formulas $Y_0(x)$, $Y_1(x)$ such that:

(a) *Both $Y_0(x)$ and $Y_1(x)$ are Yablo formulas in $KF + Cons$.*

(b) $KF + Cons \vdash Y_0(n)$; $KF + Cons \vdash \neg Y_1(n)$

Proof.

Let n be fixed; let L be the liar sentence. Define:

- $Y_0(x) := x = n \vee (x > n \wedge L)$
- $Y_1(x) := x = n + 1 \vee (x > n + 1 \wedge L)$

Then (b) is trivially satisfied. The proof of (a) (for $Y_0(x)$) is done by analyzing cases: for a fixed x, either $x < n$, or $x \geq n$. In the first case both sides of the Yablo equivalence are provably false; in the second both of them are provably true. $\square$

Yablo formulas in $KF + Compl$

Observation

Let $Y(x)$ be such that $KF + Compl \vdash Y(x) \equiv \forall z > x\, \neg T(Y(z))$. Then $KF + Compl \vdash \forall x \neg Y(x)$.

Proof.

Work in $KF + Compl$.

1. Assume $Y(x)$, so: $\forall z > x\, \neg T(Y(z))$,
2. Therefore $\forall z > x + 1\, \neg T(Y(z))$, so $Y(x + 1)$, but also $\neg T(Y(x + 1))$.
3. Since $KF + Compl$ proves **(T-in)**, we obtain $T(Y(x + 1))$ - a contradiction.

□

Yablo formulas in *KF*

Theorem

Let $Y(x)$ be such that $KF \vdash Y(x) \equiv \forall z > x \neg T(Y(z))$. Then $KF + Cons \vdash \forall x Y(x)$.

Proof (idea).

- Fix $(M, T) \models KF + Cons$. (Then $M^d \models KF + Compl$.)
- Assuming $(M, T) \models \neg Y(a)$, fix $b >_M a$ such that $(M, T) \models T(Y(b))$.
- Show that $\forall z >_M b Y(z) \notin T^d$, which means that $M^d \models Y(b)$.
- It follows that (a) $M^d \models Y(b + 1)$ and also that (b) $M^d \models \neg T(Y(b + 1))$.
- Since **(T-in)** is valid in M^d, from (a) we obtain: $M^d \models T(Y(b + 1))$, which contradicts (b).

$\square$

Independence of Yablo's sentences

We have also:

Theorem

If $Y(x)$ is a Yablo formula in KF, then $KF + \textsc{Cons} \vdash \forall z\, \neg T(Y(z))$.

We obtain the following corollaries:

Corollary

If $Y(x)$ is a Yablo formula in KF, then
$KF + \textsc{Cons} \vdash \forall x[Y(x) \equiv \neg T(Y(x))]$.

Corollary

Let $Y(x)$ be a Yablo formula in KF. Then $KF \nvdash \exists x Y(x)$ and
$KF \nvdash \exists x \neg Y(x)$.

It follows that each sentence $Y(n)$ is independent of *KF*.

Equivalence of Yablo sentences

Theorem

Let $Y(x)$ be a Yablo formula in KF such that for every $(M, T) \vDash KF$ we have:

$$\forall a \in M\, [M^* \vDash_{sk} Y(a) \text{ iff } M^* \vDash_{sk} \forall z > a\, \neg T(Y(z))].$$

Then $KF \vdash \forall xy\, [Y(x) \equiv Y(y)]$.

In the proof the properties of partial models generated by (classical) models of *KF* are heavily used.

Summary

1. All Yablo sentences are provably equivalent in *FS*; they are also provably equivalent to the statements of their own untruth.
2. *KF* with the completeness axiom proves (uniformly) negations of all sentences which are Yablo in $KF + \textsc{Compl}$.
3. In *KF* with the consistency axiom, properties of formulas which are Yablo in $KF + \textsc{Cons}$ depend on the choice of a Yablo formula. However, $KF + \textsc{Cons}$ proves (uniformly) all Yablo sentences which are Yablo in *KF*. Moreover, such sentences are provably equivalent to statements of their own untruth.
4. *KF* doesn't decide sentences which are Yablo in *KF*. However, *KF* proves the equivalence of Yablo sentences which are well behaved in partial models.

References

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THE END

Thanks for your attention!!!