T-equivalences for positive sentences

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Disquotational theories of truth can be based on the local or the uniform T-schema.

(Tr-local) \[ \text{Tr}(\Box \varphi) \equiv \varphi \]

(Tr-uniform) \[ \forall x_1 \ldots x_n [\text{Tr}(\Box \varphi(x_1 \ldots x_n)) \equiv \varphi(x_1 \ldots x_n)] \]

Disquotational axioms are then defined as all formulas obtained from (Tr-local) or (Tr-uniform) by substituting for \( \varphi \) formulas (possibly with the truth predicate) forming an appropriate recursive substitution class.
In what follows the following notation will be used:

- $L_{PA}$, $Sent_{PA}$ - arithmetical formulas and sentences.
- $L_{Tr}$, $Sent_{Tr}$ - formulas and sentences of the language of arithmetic extended with “Tr”.
- $L_{Tr}^{+}$, $Sent_{Tr}^{+}$ - positive formulas and sentences
- $Ind_{\varphi}$ - induction for a formula $\varphi$
Basic variants of disquotational theories

**Definition 1**

- \( TB(\text{PA}) = \text{PA} \cup \{ \text{Tr}(\neg \varphi) \equiv \varphi : \varphi \in L_{\text{PA}} \} \cup \text{Ind}_{L_{\text{Tr}}} \)
- \( UTB(\text{PA}) = \text{PA} \cup \{ \forall x_1 \ldots x_n [\text{Tr}(\neg \varphi(x_1 \ldots x_n)) \equiv \varphi(x_1 \ldots x_n)] : \varphi \in L_{\text{PA}} \} \cup \text{Ind}_{L_{\text{Tr}}} \)

**Fact 2**

Both \( TB(\text{PA}) \) and \( UTB(\text{PA}) \) are conservative extensions of PA. Both theories are also truth-theoretically weak.
Reactions

**Arithmetical weakness**

- Conservativeness is a desirable property of truth theories.
- Our notion of truth, even introduced via disquotational axioms, can be used in proving new arithmetical theorems (in fact arithmetical strength is desirable).
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**Truth-theoretic weakness**

- Truth-theoretic strength is not really required.
- The main point of having the notion of truth is being able to prove truth-involving generalizations.
Reactions

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The theory PUTB

Definition 3

A formula $\varphi$ of the language $L_{Tr}$ is positive iff every occurrence of “$Tr$” in $\varphi$ lies within a scope of even number of negations.

PUTB is a theory with full induction, taking as axioms all positive substitutions of (Tr-uniform)

Theorem 4

PUTB is arithmetically equivalent with KF. In particular, the truth predicate of KF is definable in PUTB.

Definition 5

\[ PTB = PA \cup \{ \text{Tr}(\neg \varphi) \equiv \varphi : \varphi \in \text{Sent}^+_\text{Tr} \} \cup \{ \text{Ind}_\varphi : \varphi \in L_{\text{Tr}} \}. \]

Theorem 6

PTB is conservative over PA.
A set of formulas $p(x, a)$ with a parameter $a$ is a type over a model $M$ iff every finite subset of $p(x, a)$ is realized in $M$.

A model $M$ is recursively saturated iff all recursive types over $M$ are realized.

Every model $M$ has a recursively saturated elementary extension of the same cardinality.
We show that:

\((\ast)\) For an arbitrary finite \(Z \subseteq PTB\) and for an arbitrary recursively saturated model \(M\), \(M\) can be extended to a model of \(L_{Tr}\) in such a way as to make all sentences in \(Z\) true.

Then (for \(\psi \in L_{PA}\)): if \(PTB \vdash \psi\), then for some finite \(Z \subseteq PTB\), \(Z \vdash \psi\); therefore by \((\ast)\), \(PA \vdash \psi\).
We define a translation function \( t(a, \varphi) \) - for \( \varphi \) belonging to \( L_{Tr} \), it gives as value an arithmetical formula with a parameter \( a \).

- \( t(a, \neg t = s) = \neg t = s \)
- \( t(a, Tr(t)) = t \in a \)
- \( t(a, \neg \psi) = \neg t(a, \psi) \), similarly for conjunction and disjunction
- \( t(a, \exists x \psi) = \exists x t(a, \psi) \), similarly for the general quantifier.
### Fact 8

Let \( d \in M \). Let \( K = (M, T) \) with \( T = \{ a : M \models a \in d \} \). Then for every \( \varphi \in L_{Tr} \), for every valuation \( v \) in \( M \), we have:

\[
M \models t(d, \varphi)[v] \iff K \models \varphi[v]
\]

The proof is by induction on the complexity of \( \varphi \). If e.g. 
\( \varphi = Tr(t) \), then we have: 
\[
M \models t(d, Tr(t))[v] \iff M \models t \in d[v] \iff val^M(t, v) \in T \iff K \models Tr(t)[v].
\] 
The proof of the other clauses is routine.
**Fact 9**

Let $M_1 = (M, A)$, $M_2 = (M, B)$ with $A, B$ being subsets of $M$ such that $A \subseteq B$. Then for every valuation $v$ in $M$, for every $\varphi(x_1\ldots x_n) \in L_{Tr}^+$, we have: if $M_1 \models \varphi(x_1\ldots x_n)[v]$, then $M_2 \models \varphi(x_1\ldots x_n)[v]$.

The proof consists in showing that every formula in $L_{Tr}^+$ is logically equivalent with some strictly positive formula, i.e. a formula in which no occurrence of “$Tr$” is negated. Then it is enough to prove by induction that every strictly positive formula satisfies the above condition.
Definition 10

Given a recursively saturated model $M$, we define a family of recursive types over $M$, a family of elements realizing these types and a family of models $M_n$ which extend $M$ to a model of $L_{Tr}$.

1. $p_0(x) = \{ \varphi \in x \equiv \varphi : \varphi \in Sent_{PA} \} \cup \{ \forall w (w \in x \Rightarrow w \in Sent_{PA}) \}$
   - $d_0$ realizes $p_0(x)$
   - $T_0 = \{ a : M \models a \in d_0 \}$
   - $M_0 = (M, T_0)$

2. $p_{n+1}(x, d_n) = \{ \varphi \in x \equiv t(d_n, \varphi) : \varphi \in Sent_{Tr}^+ \} \cup \{ \forall z (z \in d_n \Rightarrow z \in x) \} \cup \{ \forall z (z \in x \Rightarrow z \in Sent_{Tr}^+) \}$
   - $d_{n+1}$ realizes $p_{n+1}(x, d_n)$
   - $T_{n+1} = \{ a : M \models a \in d_{n+1} \}$
   - $M_{n+1} = (M, T_{n+1})$
Proof of conservativeness theorem

Observation

For every \( n \), a type \( p_n \), a model \( M_n \) and an element \( d_n \) are well defined. We have also:

\[
\forall \varphi \in \text{Sent}_{Tr} \forall n [M \models t(d_n, \varphi) \text{ iff } M_n \models \varphi].
\]
Proof of conservativeness theorem

Let $Z$ be a finite subset of $PTB$. Given a recursively saturated model $M$, we will find an $L_{Tr}$-extension of $M$ which makes $Z$ true. Let $A = \{ Tr(\neg \varphi_0) \equiv \varphi_0 \ldots \, Tr(\neg \varphi_k) \equiv \varphi_k \}$ be a set of all $T$-sentences in $Z$. Fix $n$ as the smallest natural number such that:

$$\forall i \leq k[M_n \models \varphi_i \lor \neg \exists l \in NM_l \models \varphi_i]$$

The existence of such a number follows from Fact 9 together with the observation that $T_0 \subseteq T_1 \subseteq T_2 \ldots$. Then we observe that $M_{n+1} \models Z$. Since $T_{n+1}$ is parametrically definable in $M$, it is inductive. We have also:

$$\forall i \leq kM_{n+1} \models Tr(\neg \varphi_i) \equiv \varphi_i.$$
Comment 1. All models $M_n$ satisfy the condition “$Tr(\psi) \Rightarrow \psi$” for all $\psi \in L_{Tr}$, so the same proof establishes conservativeness of a theory containing not only true-positive biconditionals with induction, but also all instances (not just the positive ones) of the “Tr-out” schema.

Comment 2. A slightly modified construction gives a proof of a stronger result (in the formulation below $\vec{z}$ stands for a sequence of variables).
Theorem 11

\( PTB \cup \{ \forall \vec{z}[\text{Tr}(\varphi(\vec{z})) \Rightarrow \varphi(\vec{z})] : \varphi(\vec{z}) \in L_{\text{Tr}} \} \) is conservative over PA.

The proof involves a different characterization of the set of types. Fixing a model \( M \) and a nonstandard \( a \in M \), we put:

- \( \rho_0(x, a) = \{ \forall \vec{z} < a[\varphi(\vec{z}) \in x \equiv \varphi(\vec{z})] : \varphi(\vec{z}) \in L_{\text{PA}} \} \cup \{ \forall w[w \in x \Rightarrow \exists \varphi(\vec{z}) \in L_{\text{PA}} \exists \vec{s} < a w = \lceil \varphi(\vec{s}) \rceil] \}

- \( \rho_{n+1}(x, d_n, a) = \{ \forall \vec{z} < a[\varphi(\vec{z}) \in x \equiv t(d_n, \varphi(\vec{z}))] : \varphi(\vec{z}) \in L_{\text{Tr}}^+ \} \cup \{ \forall z[z \in d_n \Rightarrow z \in x] \cup \{ \forall w[w \in x \Rightarrow \exists \varphi(\vec{z}) \in L_{\text{Tr}}^+ \exists \vec{s} < a w = \lceil \varphi(\vec{s}) \rceil] \}

with \( d_n \) and \( M_n \) defined exactly as before.