T-equivalences for positive sentences

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Disquotational truth

Disquotational theories of truth can be based on the local or the uniform T-schema.

(Tr-local) $$Tr(\ulcorner \varphi \urcorner) \equiv \varphi$$

(Tr-uniform) $$\forall x_1 \ldots x_n [Tr(\ulcorner \varphi(x_1 \ldots x_n) \urcorner) \equiv \varphi(x_1 \ldots x_n)]$$

Disquotational axioms are then defined as all formulas obtained from (Tr-local) or (Tr-uniform) by substituting for φ formulas (possibly with the truth predicate) forming an appropriate recursive substitution class.

Notation

In what follows the following notation will be used:

- L_{PA}, $Sent_{PA}$ - arithmetical formulas and sentences.
- L_{Tr}, $Sent_{Tr}$ - formulas and sentences of the language of arithmetic extended with "*Tr*".
- L^{+}_{Tr}, $Sent^{+}_{Tr}$ - positive formulas and sentences
- Ind_{φ} - induction for a formula φ

Definition 1

- $TB(PA) = PA \cup \{Tr(\ulcorner\varphi\urcorner) \equiv \varphi : \varphi \in L_{PA}\} \cup Ind_{L_{Tr}}$
- $UTB(PA) = PA \cup \{\forall x_1 ... x_n [Tr(\ulcorner\varphi(x_1 ... x_n)\urcorner) \equiv \varphi(x_1 ... x_n)] : \varphi \in L_{PA}\} \cup Ind_{L_{Tr}}$

Fact 2

Both $TB(PA)$ and $UTB(PA)$ are conservative extensions of PA. Both theories are also truth-theoretically weak.

Arithmetical weakness

- Conservativeness is a desirable property of truth theories.
- Our notion of truth, even introduced via disquotational axioms, can be used in proving new arithmetical theorems (in fact arithmetical strength is desirable).

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Truth-theoretic weakness

- Truth-theoretic strength is not really required.
- The main point of having the notion of truth is being able to prove truth-involving generalizations

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Definition 3

- A formula φ of the language L_{Tr} is positive iff every occurrence of "*Tr*" in φ lies within a scope of even number of negations.
- *PUTB* is a theory with full induction, taking as axioms all positive substitutions of (Tr-uniform)

Theorem 4

PUTB is arithmetically equivalent with KF. In particular, the truth predicate of KF is definable in PUTB.

Source: Halbach,V. "Reducing compositional to disquotational truth", *The Review of Symbolic Logic* (2009), 2: 786-798.

Definition 5

$PTB = PA \cup \{Tr(\ulcorner\varphi\urcorner) \equiv \varphi : \varphi \in Sent^{+}_{Tr}\} \cup \{Ind_{\varphi} : \varphi \in L_{Tr}\}$.

Theorem 6

PTB is conservative over PA.

- A set of formulas $p(x, a)$ with a parameter a is a type over a model M iff every finite subset of $p(x, a)$ is realized in M.
- A model M is recursively saturated iff all recursive types over M are realized.
- Every model M has a recursively saturated elementary extension of the same cardinality.

We show that:

(∗) For an arbitrary finite $Z \subseteq PTB$ and for an arbitrary recursively saturated model M, M can be extended to a model of L_{Tr} in such a way as to make all sentences in Z true.

Then (for $\psi \in L_{PA}$): if $PTB \vdash \psi$, then for some finite $Z \subseteq PTB,\ Z \vdash \psi$; therefore by (∗), $PA \vdash \psi$.

Definition 7

We define a translation function $t(a, \varphi)$ - for φ belonging to L_{Tr}, it gives as value an arithmetical formula with a parameter a.

- $t(a, \ulcorner t = s \urcorner) = \ulcorner t = s \urcorner$
- $t(a, Tr(t)) = \ulcorner t \in a \urcorner$
- $t(a, \neg\psi) = \neg t(a, \psi)$, similarly for conjunction and disjunction
- $t(a, \exists x \psi) = \exists x t(a, \psi)$, similarly for the general quantifier.

Fact 8

Let $d \in M$. Let $K = (M, T)$ with $T = \{a : M \models a \in d\}$. Then for every $\varphi \in L_{Tr}$, for every valuation v in M, we have:

$$M \models t(d, \varphi)[v] \textit{ iff } K \models \varphi[v]$$

The proof is by induction on the complexity of φ. If e.g. $\varphi = Tr(t)$, then we have: $M \models t(d, Tr(t))[v]$ iff $M \models t \in d[v]$ iff $val^M(t, v) \in T$ iff $K \models Tr(t)[v]$. The proof of the other clauses is routine.

Fact 9

Let $M_1 = (M, A), M_2 = (M, B)$ *with* A, B *being subsets of* M *such that* $A \subseteq B$. *Then for every valuation* v *in* M, *for every* $\varphi(x_1 ... x_n) \in L^+_{Tr}$, *we have: if* $M_1 \models \varphi(x_1 ... x_n)[v]$, *then* $M_2 \models \varphi(x_1 ... x_n)[v]$.

The proof consists in showing that every formula in L^+_{Tr} is logically equivalent with some *strictly positive* formula, i.e. a formula in which no occurrence of "*Tr*" is negated. Then it is enough to prove by induction that every strictly positive formula satisfies the above condition.

Definition 10

Given a recursively saturated model M, we define a family of recursive types over M, a family of elements realizing these types and a family of models M_n which extend M to a model of L_{Tr}.

1.
   - $p_0(x) = \{\varphi \in x \equiv \varphi : \varphi \in Sent_{PA}\} \cup \{\forall w(w \in x \Rightarrow w \in Sent_{PA}\}$
   - d_0 realizes $p_0(x)$
   - $T_0 = \{a : M \models a \in d_0\}$
   - $M_0 = (M, T_0)$
2.
   - $p_{n+1}(x, d_n) = \{\varphi \in x \equiv t(d_n, \varphi) : \varphi \in Sent^{+}_{Tr}\} \cup \{\forall z(z \in d_n \Rightarrow z \in x)\} \cup \{\forall z(z \in x \Rightarrow z \in Sent^{+}_{Tr})\}$
   - d_{n+1} realizes $p_{n+1}(x, d_n)$
   - $T_{n+1} = \{a : M \models a \in d_{n+1}\}$
   - $M_{n+1} = (M, T_{n+1})$

Observation

For every n, a type p_n, a model M_n and an element d_n are well defined. We have also:

$$\forall \varphi \in Sent_{Tr} \forall n \, [M \models t(d_n, \varphi) \text{ iff } M_n \models \varphi].$$

Let Z be a finite subset of PTB. Given a recursively saturated model M, we will find an L_{Tr}-extension of M which makes Z true. Let $A = \{Tr(\ulcorner\varphi_0\urcorner) \equiv \varphi_0 \ldots Tr(\ulcorner\varphi_k\urcorner) \equiv \varphi_k\}$ be a set of all T-sentences in Z. Fix n as the smallest natural number such that:

$$\forall i \leq k[M_n \models \varphi_i \vee \neg\exists l \in N M_l \models \varphi_i]$$

The existence of such a number follows from Fact 9 together with the observation that $T_0 \subseteq T_1 \subseteq T_2 \ldots$. Then we observe that $M_{n+1} \models Z$. Since T_{n+1} is parametrically definable in M, it is inductive. We have also:

$$\forall i \leq k M_{n+1} \models Tr(\ulcorner\varphi_i\urcorner) \equiv \varphi_i.$$

Additional comments

Comment 1. All models M_n satisfy the condition "$Tr(\psi) \Rightarrow \psi$" for all $\psi \in L_{Tr}$, so the same proof establishes conservativeness of a theory containing not only true-positive biconditionals with induction, but also all instances (not just the positive ones) of the "Tr-out" schema.

Comment 2. A slightly modified construction gives a proof of a stronger result (in the formulation below $\vec{z}$ stands for a sequence of variables).

Theorem 11

$PTB \cup \{\forall \vec{z}[Tr(\varphi(\vec{z})) \Rightarrow \varphi(\vec{z})] : \varphi(\vec{z}) \in L_{Tr}\}$ is conservative over PA.

The proof involves a different characterization of the set of types. Fixing a model M and a nonstandard $a \in M$, we put:

- $p_0(x, a) = \{\forall \vec{z} < a[\varphi(\vec{z}) \in x \equiv \varphi(\vec{z})] : \varphi(\vec{z}) \in L_{PA}\} \cup \{\forall w[w \in x \Rightarrow \exists \varphi(\vec{z}) \in L_{PA} \exists \vec{s} < a\ w = \ulcorner \varphi(\vec{s}) \urcorner\}$
- $p_{n+1}(x, d_n, a) = \{\forall \vec{z} < a[\varphi(\vec{z}) \in x \equiv t(d_n, \varphi(\vec{z}))] : \varphi(\vec{z}) \in L^{+}_{Tr}\} \cup \{\forall z[z \in d_n \Rightarrow z \in x\} \cup \{\forall w[w \in x \Rightarrow \exists \varphi(\vec{z}) \in L^{+}_{Tr} \exists \vec{s} < a\ w = \ulcorner \varphi(\vec{s}) \urcorner\}$

with d_n and M_n defined exactly as before.

References

- Cezary Cieśliński ‘T-equivalences for positive sentences’, ‘The Review of Symbolic Logic’ 4(2), 319–325, 2011.
- Cezary Cieśliński ‘Typed and Untyped Disquotational Truth’, in *Unifying the Philosophy of Truth*, T. Achourioti and H. Galinon and K. Fujimoto and J. Martínez-Fernández (eds), *Logic, Epistemology, and the Unity of Science* vol. 36, 307–320 Springer, Dordrecht, 2015.
- Cezary Cieśliński *The Epistemic Lightness of Truth. Deflationism and its Logic*, Cambridge University Press, 2017.